Optimization Based Passive Constrained Fitting

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ABSTRACT

Verification of contemporary integrated circuits requires accurate modeling of high-frequency effects in all passive component subsystems. Often, descriptions of those subsystems are only available in the frequency-domain. In this paper, we propose a simple, scalar, constrained-passive rational approximation scheme that incorporates a grid-based test for strict positive realness. In contrast to similar recent work based on convex optimization and the positive real lemma, the methodology is potentially more efficient, since it does not introduce a quadratic number of auxiliary variables, is potentially more accurate, since pole locations can be re-adjusted during the optimization, but possibly less reliable, since it relies on the solution of optimization problems that are not convex. Therefore, because of the use of local constrained non-convex optimization, the generation of feasible initial guesses is also considered.

1. INTRODUCTION

Full verification of the correct functionality of some of the complex systems currently being designed, such as RF front-ends, usually requires simulation at various levels including, invariably, time-domain simulation of the system. Unfortunately, it is not a trivial matter to generate accurate circuit-level models for all the devices or modules used in the design. Nevertheless, such models are necessary for the time domain simulation of larger designs and to account for many of the non-ideal characteristics of the devices or modules. In high-frequency applications in particular, certain devices are usually, and often exclusively, described and studied in the frequency domain. Devices such as coil inductors, SAW filters, non-ideal transmission lines and high-frequency transistors are commonly described by manufacturers and designers by their frequency dependent scattering parameter or admittance matrices. An attractive approach to modeling such devices in time-domain simulators is the generation of state-space models via rational approximation. It is necessary that these models accurately represent the original subsystems, i,e., that they have frequency responses that match the available data but, they must also possess intrinsic properties similar to those of the physical system that they represent, in particular, passivity.

For the purposes of this work we assume as a starting point that frequency domain data describing a subsystem is available, having been obtained either through measurement or through physical simulation. In either case, the available data is sampled, incomplete, noisy, and covers only a finite range of the spectrum. Physically, the data to be approximated corresponds to the impedance or admittance functions of passive linear systems. If only scattering parameter data is available, it can be transformed to the corresponding admittance or impedance, (see [8] for details).

Several algorithms for stable rational approximation exist and can be used for frequency domain identification of stable systems, for example [13, 14, 15, 9, 6] and the references therein. Recently, in [7], a convex programming formulation for the positive real constrained rational approximation problem was presented. In this work, after an initial guess is generated, the denominator of the rational function approximation is kept fixed while the numerator was recalculated in order to improve accuracy while maintaining system properties. This approach was based on a state space formulation of the system and used the Positive Real Lemma [1, 4], and convex optimization techniques [16] to generate a guaranteed passive interpolant. However, due to the introduction of auxiliary variables in the positive real lemma, the high computational cost of this procedure make it inappropriate for approximation with high-order system models. In addition, it assumed a set of poles that was fixed and unchanged in the final optimization stage, where passivity was imposed.

In this paper we present a formulation that allows the simultaneous optimization of the numerator and denominator coefficients of a rational model of a system described in the frequency domain. This task is accomplished by using a different characterization of strictly positive real functions [2]. This alternative characterization allows us to replace the matrix inequalities associated with the positive real lemma with a set of simple scalar inequalities evaluated on a grid. Ideally, the size of this grid should vary linearly with the order of the model. Therefore, the number of unknowns in the optimization process should grow linearly with the model order instead of quadratically as in [7]. Additionally, since we are optimizing both the numerator and denominator coefficients simultaneously, we may be able to achieve higher accuracy with a given order model. However, since the optimization problem is no longer convex, care must be taken with issues such as generation of an appropriate initial guess. In this paper only the scalar, single-

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input/single-output (SISO), formulation will be discussed. However, in Section 2 we present an extension to the main theoretical result in [2] which may prove promising in extending the algorithm to multi-input, multi-output (MIMO) systems.

The paper is structured as follows. First, a brief review on passivity and the main theoretical basis for this work, namely the scalar positive real characterization from [2], is presented. Then, an iterative positive real constrained rational approximation algorithm is proposed for generating time-domain models by approximating a system's frequency domain characteristics. Since the algorithm is formulated as an optimization problem which requires the construction of a grid of points satisfying a specific set of properties, issues such as initial point generation as well as grid construction and updating are then considered. The proposed algorithm is then applied to a set of real world examples, and a brief comparison with the work in [7] is performed.

2. BACKGROUND

2.1 Passivity and Positive-Realness

An important sub-class of physical systems are passive systems, those systems that do not generate energy. Most linear elements encountered in circuit analysis, such as networks composed of resistors, capacitors and inductors, interconnect networks, and circuit packages, are passive, and often strictly passive, meaning that they always consume energy. Passivity of individual elements implies strong constraints on the behavior of composite networks, as interconnected (strictly) passive systems are (strictly) passive in turn. Stable systems do not possess this closure property. Stable systems loaded by stable, and even passive systems, may not constitute an overall stable system. Therefore, it is important that models of passive components themselves be passive, or anomalous behavior may be expected in time-domain simulations.

The admittance and impedance parameter matrices of passive electrical networks are positive real matrix rational functions [5]. A matrix function $\mathbf{H}(s)$ is positive real if,

$$\mathbf{H}(s) = \overline{\mathbf{H}(\overline{s})} \quad (1)$$

$$\mathbf{H}(s) \text{ is analytic in } Re[s] > 0 \quad (2)$$

$$\mathbf{H}(s) + \mathbf{H}(s)^H \geq 0 \text{ in } Re[s] > 0. \quad (3)$$

In essence these conditions imply that the systems must have a "positive resistivity". A matrix rational function is positive real if and only if (1) and (2) hold and

$$\mathbf{H}(j\omega) + \mathbf{H}(j\omega)^H \geq 0 \text{ for } \omega \in \mathbb{R}. \quad (4)$$

Any pole on the imaginary axis must be simple and its residue matrix must be Hermitian and nonnegative definite. Note that, if $\mathbf{H}(s)$ has no poles on the closed right-half plane, it is positive real if and only if (4) holds. In model generation and approximation, it is necessary to have a condition for positive-realness that involves the variables to be analyzed or optimized. For example, the matrix transfer function of a linear time-invariant system is positive real if its state space realization satisfies the Positive Real Lemma [1, 3]. In this work, we shall show that alternative conditions can have advantages for optimization-based approaches.

In the following subsection, we present a test for the strict positive-realness of a *scalar* rational function that requires the evaluation of a finite set of simple constraints. This test is the basis for the constrained optimization algorithm presented in Section 3. Although the test and the proposed algorithm are limited to the scalar case, an extension of the test to the multivariable case is proposed and should allow, in the future, the passive constrained approximation algorithm to deal with multivariable systems.

2.2 A Test for Strict Positive Realness

Let us then restrict our attention to the *scalar*, single-input/single-output, problem, where $\mathbf{H}(j\omega) : \mathbb{R} \to \mathbb{C}$. In [2] a set of necessary and sufficient conditions for the strict positive realness of a scalar rational function was presented. In contrast to the matrix inequalities of the positive real lemma, these conditions are simple pointwise or pairwise phase constraints on a finite grid of $N+1$ points where $\omega_k < \omega_{k+1}$, $\omega_0 = 0$ and $\omega_N = \infty$. We chose to use this result in our formulation because of the simplicity and compactness of the constraints. In the following we adopt the convention that $\arg[p(j\omega_1)/p(j\omega_2)] \in [-\pi, \pi]$.

THEOREM 1 (ANDERSON-MANSOUR-KRAUS [2]). *Let $p(s)$ and $q(s)$ be real polynomials of degree n, with positive leading coefficients. Then $p(s)/q(s)$ is strictly positive real if and only if there exist frequencies $\omega_0 = 0 < \omega_1 < \ldots < \omega_N = \infty$ such that equations (5) through (10) are satisfied.*

$$0 < \arg \frac{p(j\omega_k)}{p(j\omega_{k-1})} \leq \frac{\pi}{4} \quad (5) \qquad 0 < \arg \frac{q(j\omega_k)}{q(j\omega_{k-1})} \leq \frac{\pi}{4} \quad (6)$$

$$\sum_{k=1}^{N} \arg \frac{p(j\omega_k)}{p(j\omega_{k-1})} = \frac{n\pi}{2} \quad (7) \qquad \sum_{k=1}^{N} \arg \frac{q(j\omega_k)}{q(j\omega_{k-1})} = \frac{n\pi}{2} \quad (8)$$

$$-\frac{\pi}{2} < \arg \frac{p(j\omega_k)}{q(j\omega_{k-1})} < \frac{\pi}{2} \quad (9) \qquad -\frac{\pi}{2} < \arg \frac{p(j\omega_{k-1})}{q(j\omega_k)} < \frac{\pi}{2} \quad (10)$$

For a detailed proof and the discrete time version of the theorem see [2].

From the theorem we can easily see that when the degrees of $p(s)$ and $q(s)$ are not equal, they may only differ by 1. In that case, however, the strict inequalities in (10) and (9), must be replaced by non strict inequalities, for $k = N$.

Theorem 1 underlines a test for positive-realness of a scalar function. The test hinges on the existence and construction of a grid satisfying conditions (5) through (10). In order for there to be a grid where (6) and (8) can be satisfied, it is necessary that the roots of the denominator be in the open left-half plane. This can be verified by noting that, if any root were in the right-half plane then its phase contribution would be $-\pi/2$ and (8) would not be satisfied. Moreover, if any root were to lie on the imaginary axis, (6) could not be satisfied because the phase shift at that point would be $-\pi$. The same reasoning applies to (5) and (7), which impose that all the roots of the numerator also lie in the open left-half plane. Equations (9) and (10) are *shifted* scalar versions of condition (4), $\mathrm{Re}[p(\omega)/q(\omega)] > 0$, which physically is equivalent to requiring that the model's resistivity or conductivity, be positive. Together with the grid density conditions, Eqns. (5) and (6), these *shifted* equations, each of which couples the points ω_k and ω_{k+1}, guarantee that $\mathrm{Re}[p(\omega)/q(\omega)] > 0$ in each $[\omega_k, \omega_{k+1}]$ interval. Since the grid spans the whole axis and there are no roots of $p(s)$ or $q(s)$ on the closed right half plane, the rational function is strictly positive real. One of the strengths of this result is its simple interpretation and the fact that only scalar conditions are used. Furthermore, it relates the problem of generating a positive-real transfer function into that of constructing a grid of points satisfying Eqns (5) to (10). Theorem 1 applies to the scalar case. While we will not provide algorithms for MIMO systems in this paper, we mention that often scalar positivity conditions can be extended to the MIMO case, as in the following result:

THEOREM 2 (MULTIVARIABLE POSITIVE-REALNESS TEST). *Given an $m \times m$ multivariable proper rational function $\mathbf{H}(s)$ and $\mathbf{G}(s) = \mathbf{H}(s) + \mathbf{H}^H(s)$. The matrix rational function $\mathbf{H}(s)$ is strictly positive real if $\mathbf{G}(0)$ is positive definite and the scalar rational function $g(s) = \det\left(\mathbf{H}(s) + \mathbf{H}^H(s)\right)$ is strictly positive real.*

PROOF. If $g(s)$ is strictly positive real, $g(s)$ has no poles in the closed right half plane and $g(j\omega) > 0$ for all $\omega \in \mathbb{R}$. Since $\mathbf{G}(s)$ is Hermitian, all its eigenvalues are real. Moreover, because $g(j\omega)$ is analytic for all $\omega \in \mathbb{R}$, the eigenvalues of $\mathbf{G}(j\omega)$ are continuous functions of ω. Therefore, if $g(j\omega) > 0$ each eigenvalue $\lambda_k[G(j\omega)]$ is either positive or negative for all $\omega \in \mathbb{R}$. If $\mathbf{G}(0)$ is positive definite, all its eigenvalues are positive and so $\mathbf{H}(s)$ is strictly positive real. □

Therefore, in principle, a MIMO strict positive realness test can be performed by testing that $\mathbf{G}(0)$ is positive and that the scalar rational function $g(s)$ is strictly positive real which can be accomplished applying Theorem 1. We stress that while this extension is proposed, the resulting discussion and all the results in this paper refer to the scalar case and are thus based directly on Theorem 1. We include Theorem 2 for its novelty, for completeness and as a direction for further work.

3. CONSTRAINED FITTING ALGORITHM

By using Theorem 1, the problem of generating guaranteed passive models for frequency described data can be formulated as an optimization problem. Armed with an appropriate grid generation scheme (presented later in Section 4) and a constrained non-linear optimization solver, a simple algorithm for generating strictly positive-real scalar rational approximations can thus be devised.

Since we are solving for both the numerator and denominator coefficients, the optimization problem is not convex and, therefore, care must be taken in order to generate a positive real initial guess. Therefore, the algorithm starts by generating an initial model that is positive real, after which a grid satisfying the conditions of Theorem 1 is generated using the algorithm presented in Section 4. The model is then iteratively improved. At each step, the grid is adapted and a constrained optimization problem is solved, minimizing the error between the data and the model at a set of given data points while satisfying (5) through (10). The algorithm stops when no further improvement is made. The overall flow of the algorithm is shown as Algorithm 1. The issue of initial guess generation is discussed in Section 3.1. The constrained optimization problem is formulated and discussed in Section 3.2. Finally the grid generation and updating algorithm is the subject of Section 4.

3.1 Initial Guess Generation

To begin the optimization procedure, it is necessary to obtain a feasible initial guess. This means we must find a (probably not optimal) rational function that satisfies the constraints of Theorem 1. In this work, for purposes of initial-guess generation only, we assume a form that decomposes the rational function into a sum of low-order rational functions, each of which is constrained to be positive-real. By requiring that each rational function be positive real, the approximant is guaranteed to be positive real. For rational functions of low enough order there exist simple algebraic constraints that guarantee their positive realness [10]. When the rational function is represented as a sum of first and second order rational functions,

$$H(s) = \sum_{k=1}^{k_{max}} \frac{b_1^{(k)} s + b_0^{(k)}}{s^2 + a_1^{(k)} s + a_0^{(k)}} + \sum_{r=1}^{r_{max}} \frac{b_2^{(r)}}{s + a_2^{(r)}}, \qquad (11)$$

Algorithm 1 Phase constrained fitting algorithm

Generate starting point with positive real frequency response.
repeat
  Adapt grid.
  Apply constrained optimization problem solver for a fi xed number of iterations using the current model as initial guess.
until Improvement stops.

then each term in (11) is positive real if and only if $b_2^{(r)} > 0$, $a_2^{(r)} > 0$, $b_1^{(k)} > 0$, $b_0^{(k)} > 0$, and $b_0^{(k)} - b_1^{(k)} a_1^{(k)} > 0, 1 \le k \le K_{max}$. To obtain such a form, we assume that a set of stable poles is available, such as from the procedures in [9, 11]. These provide the denominator coefficients in (11). Once the denominator coefficients are fixed, the remaining constraints are merely linear restrictions on the numerator coefficients. Therefore, the problem of minimizing the square error between the model and the data can be formulated as a linearly constrained linear least-squares problem which is convex and thus fairly easy to solve. Alternatively, a fixed denominator algorithm using Eqns. (19) through (22) and the grid generation and updating algorithm presented in section 4 can be used, as can the algorithm proposed in [7], though this is quite expensive.

The astute reader may wonder why we simply do not base our algorithm on the decomposition of the transfer function into sums of positive-real functions, as in fact is the approach suggested in [12]. The reason is that not all positive-real rational functions can be represented as the sum of low-order rational functions, each positive-real. That is in fact a sufficient but not necessary condition for a rational function to be positive-real. A trivial counter-example can be constructed by taking two scalar positive real functions, one larger than the other uniformly in the RHP, and subtracting them. For this reason, algorithms based on summing low-order positive-real functions are always less accurate than algorithms that utilize representations with one-to-one correspondence with the full set of high-order positive-real functions, and may not be able to fit complicated data at all. In the examples in Section 5.1, we will further demonstrate the problems with using such an approach by showing its application to one of the example data sets.

3.2 Solving the Optimization Problem

The constrained optimization problem can then be presented as

$$\begin{aligned}
\text{minimize} \quad & \mathcal{E}^H \mathcal{E} && (12)\\
\text{with} \quad & \mathcal{E} = \mathbf{W}\left(H(j\omega_d) - \mathbf{H}_d\right) \\
\text{subject to} \quad & p(s) \text{ and } q(s) \text{ are real and Hurwitz} && (13)\\
& g_1(k) = \arg\left[p(\omega_k)/p(\omega_{k-1})\right] - \pi/4 < 0 && (14)\\
& g_1(k) = \arg\left[q(\omega_k)/q(\omega_{k-1})\right] - \pi/4 < 0 && (15)\\
& g_3(k) = \left|\arg\left[p(\omega_k)/q(\omega_{k-1})\right]\right| - \pi/2 < 0 && (16)\\
& g_4(k) = \left|\arg\left[p(\omega_{k-1})/q(\omega_k)\right]\right| - \pi/2 < 0 && (17)\\
& \text{for } k = 1, \ldots, N_g,
\end{aligned}$$

where ω_d represents the vector of N_d frequency points at which the data $\mathbf{H}_d$ is provided. ω_g, $\omega_g(k) = \omega_k$ represents the current vector of N_g grid points. $H(j\omega_d)$ represents the frequency response of the model evaluated at the frequency points in the vector ω_d and $\mathbf{W}$ represents an N_p by N_d error projection matrix that adds flexibility to the algorithm by allowing the choice of different error criteria. Note that, in order to guarantee that Eqns. (7) and (8) can be satisfied, we explicitly impose that the numerator and denominator polynomials be Hurwitz, i.e. that both $p(s)$ and $q(s)$ be polynomials with real positive coefficients and roots that are either negative or pairwise conjugate with negative real parts.

By representing $p(s)$ and $q(s)$ as a product of biquadratic sections,

$$H(s) = \frac{p(s)}{q(s)} = K\frac{s+b_0^{(0)}}{s+a_0^{(0)}}\prod_{k=1}^{\tilde{n}}\frac{s^2+b_1^{(k)}s+b_0^{(k)}}{s^2+a_1^{(k)}s+a_0^{(k)}}, \quad (18)$$

Eqn. (13) may be replaced by

$$a_1^{(k)} > 0, a_2^{(k)} > 0, b_1^{(k)} > 0, b_2^{(k)} > 0, K > 0, \forall k.$$

Also, since the unwrapped value of the numerator and denominator phase, represented as $\Phi_p(\omega)$ and $\Phi_q(\omega)$, are easily calculated, Eqns. (14) through (17) can be replaced by

$$g_1(k) = \Phi_p(\omega_k) - \Phi_p(\omega_{k-1}) - \pi/4 < 0 \quad (19)$$

$$g_2(k) = \Phi_q(\omega_k) - \Phi_q(\omega_{k-1}) - \pi/4 < 0 \quad (20)$$

$$g_3(k) = \left|\Phi_p(\omega_k) - \Phi_q(\omega_{k-1})\right| - \pi/2 < 0 \quad (21)$$

$$g_4(k) = \left|\Phi_p(\omega_{k-1}) - \Phi_q(\omega_k)\right| - \pi/2 < 0 \quad (22)$$

4. A GRID GENERATION ALGORITHM

In this section we present an algorithm that generates a grid satisfying Theorem 1's conditions or proves that none exists, by returning a list of frequency points at which the rational function is not positive real. This algorithm is also used to adapt the grid during the fitting process proposed in Section 3. If the theorem is to hold for each iteration of the fitting process, it is expected that the grid be adapted to the current model at each step. A grid that is finer, that has more points, than strictly necessary may be used to allow greater freedom at each step and thus reduce the number or need for grid adaptations between optimization steps. The proposed algorithm starts by generating a coarse grid such that Eqns. (5) through (8) are satisfied. In a second stage, a grid refining algorithm is applied in order to satisfy Eqns. (9) and (10). Finally, a grid pruning algorithm that reduces the redundancy in the grid is applied.

4.1 Initial Grid Generation

Let $r(s)$ represent a generic order n polynomial with real coefficients. Assuming $r(s)$ is Hurwitz, it is always possible to find a grid such that

$$0 < \arg\frac{r(j\omega_k)}{r(j\omega_{k-1})} \leq \frac{\pi}{4} \qquad \sum_{k=1}^{N}\arg\frac{r(j\omega_k)}{r(j\omega_{k-1})} = \frac{n\pi}{2}. \quad (23)$$

Without loss of generality, assume that the rational function is represented as the product of biquadratic sections from Eqn. (18) and that both $p(s)$ and $q(s)$ are odd ordered real Hurwitz polynomials. Then the unwrapped phase change of, for instance, $q(j\omega)$, from 0 to ω, which we represent by $\Phi_q(\omega)$ is given by

$$\Phi_q(\omega) = \sum_{k=1}^{\tilde{n}}\arg\left[a_0^{(k)} - \omega^2 + ja_1^{(k)}\omega\right] + \arg\left[j\omega + a_0^{(0)}\right]. \quad (24)$$

The Newton-Raphson algorithm can be used to solve for $\Phi_r(\omega_k) = k\pi/2N$ for $k = 1,\ldots,N-1$, for both $p(s)$ and $q(s)$.

4.2 Updating and Refining the Grid

The grid refining algorithm, which is also used for updating purposes, is required to find a grid satisfying Eqns. (5) through (10). Alternatively the algorithm is required to prove that $p(s)/q(s)$ is not strictly positive real by finding a counter-example, that is ω_{bad} such that $|\arg[p(j\omega_{bad})/q(j\omega_{bad})]| > \pi/2$.

Given an initial grid, the algorithm starts by ensuring that, for both the numerator and the denominator, $|\Phi_r(\widetilde{\omega}_0) - \varepsilon_1| < \varepsilon_2$ and $|\Phi_r(\widetilde{\omega}_N) - n\pi/2 + \varepsilon_1| < \varepsilon_2$. For small enough ε_1 and ε_2, this procedure guarantees that any relevant changes in the argument of the rational function occur between $\widetilde{\omega}_0$ and $\widetilde{\omega}_N$. Wherever the argument of the numerator or the denominator changes by more than $\pi/4$, new points are introduced so that Eqns. (5), (7), (6) and (8) are satisfied. While used in the iterative fitting algorithm, to avoid having to update the grid very often, the threshold $\pi/4$ may be reduced. Before proceeding, the value of the rational function is evaluated at the grid points. If the orders of the numerator and denominator are the same and

$$\left|\arg\frac{p(j\omega_k)}{q(j\omega_k)}\right| \geq \frac{\pi}{2}, \text{ or,}$$
$$\mathrm{Re}[p(j\omega_k)]\,\mathrm{Re}[q(j\omega_k)] + \mathrm{Im}[p(j\omega_k)]\,\mathrm{Im}[q(j\omega_k)] \leq 0 \quad (25)$$

for any k, the rational function is not strictly positive real. If the orders of the numerator and denominator are not the same, they may only differ by one. In these conditions $\arg[p(j\omega_N)/q(j\omega_N)]$ must be either $\pi/2$ or $-\pi/2$. At this point it remains to check if the grid satisfies Eqns. (10) and (9). If it does, the rational function is proved to be strictly positive real. Otherwise, the grid is adaptively refined until it does satisfy these conditions or (25) occurs.

Since Eqns. (9) and (10) are pairwise conditions, the grid refining process is applied locally. Let t represent a grid point at which these conditions are not satisfied. According to Theorem 1, there must be a set of points, $\widetilde{\omega}_1,\ldots,\widetilde{\omega}_M$, between ω_{t-1} and ω_t such that the sequence $\omega_{t-1},\widetilde{\omega}_1,\ldots,\widetilde{\omega}_M,\omega_t$ satisfies (9) and (10) or that $|\arg[p(j\widetilde{\omega}_k)/q(j\widetilde{\omega}_k)]| > \pi/2$ for some k. Algorithm 2 recursively builds such a grid, by halving the interval $[\omega_{t-1},\omega_t]$, and proceeding to test its midpoint $\omega_c = (\omega_{t-1}+\omega_t)/2$, as a possible additional candidate.

The proposed recursive grid refining algorithm may introduce unnecessary points. To illustrate this consider for instance a situation when, while refining the upper half of an interval we obtain another point $\widetilde{\omega}_1$ (in Algorithm 2 this would be the first element of Ω_{right}) such that the pair formed by $\widetilde{\omega}_1$ and ω_{t-1} satisfies (9) and (10). In this case, ω_c is not required. Although, it would be possible to

Algorithm 2 Recursive grid refining algorithm

$\left[\widetilde{\Omega}, code\right]$ =refi ne(...,ω_{t-1},ω_t,level)
input rational function parameterization, ω_t, ω_{t-1} and level.
output $\widetilde{\Omega} = \widetilde{\omega}_1,\ldots,\widetilde{\omega}_M$ and return code.
Determine $\omega_c = (\omega_{t-1}+\omega_t)/2$ and set $\widetilde{\Omega} = \omega_c$.
if $|\arg p(\omega_c) - \arg q(\omega_c)| \geq \pi/2$ **then**
  Indicate a point satisfying (25) was found and return.
end if
if $|\arg p(\omega_c) - \arg q(\omega_{t-1})| \geq \pi/2$ or $|\arg p(\omega_{t-1}) - \arg q(\omega_c)| \geq \pi/2$ **then**
  if $level > 0$ **then**
    $\left[\Omega_{left}, code_{left}\right]$= refi ne(...,$\omega_{t-1}$,$\omega_c$,$level-1$)
    $\widetilde{\Omega} = \left[\Omega_{left}, \widetilde{\Omega}\right]$
    Check $code_{left}$ and update current return code.
  else
    Recursion level exceeded (inconclusive).
  end if
end if
if $|\arg p(\omega_t) - \arg q(\omega_c)| \geq \pi/2$ or $|\arg p(\omega_c) - \arg q(\omega_t)| \geq \pi/2$ **then**
  if $level > 0$ **then**
    $\left[\Omega_{right}, code_{right}\right]$ =refi ne(...,ω_t,ω_t,$level-1$)
    $\widetilde{\Omega} = \left[\widetilde{\Omega}, \Omega_{right}\right]$
    Check $code_{right}$ and update current return code.
  else
    Recursion level exceeded (inconclusive).
  end if
end if
merge the grids obtained from the numerator and denominator and perform prunning in order to deal with redundancy

check for this condition in the recursive algorithm, it was decided that these points should be removed by applying some simple a-posteriori pruning algorithm that should also deal with the redundant points introduced by merging the numerator and denominator grids.

5. RESULTS

5.1 Accuracy Comparisons

In this section, we apply the proposed algorithm to four real world examples. For all the example systems, and for the purpose of determining and comparing the accuracy of the models obtained, we also applied the algorithm proposed in [7], based on a convex programming formulation for the positive real constrained rational approximation. In all cases, with either algorithm, models with a positive real transfer function that closely matches the tabulated data were obtained. In our implementation the constrained optimization problem in Algorithm 1 is solved by using `constr` from Matlab's optimization toolbox.

In our first example, a model for the $y_{1,1}$ parameter of the SMD 3-port filter model F32-680 from TDK was generated. Using the algorithm of [9], an order 10 numerator and denominator initial approximation was obtained and stabilized. The full optimization and convex formulation algorithms were then applied resulting in two passive models whose frequency behavior is displayed in Fig. 1. Again, from observation of the phase plot it can be seen that the two models are passive. The results in Fig. 1 also show that for this example, the full optimization algorithm generates a more accurate model, as can be seen by comparing the behavior of the models around the lower frequency content of the data. Note that though the two algorithms have *large* deviations outside the data range, this is acceptable. Because the original data is only specified on a finite frequency range, the only constraint outside that range is that both models be passive, and otherwise the behavior is unspecified. This example validates the use of the full optimization scheme and indicates that by manipulating both the poles and zeros of the system, accuracy improvements in the model can be obtained.

In our next example we computed models for the $y_{1,1}$ parameter of yet another SMD 3-port filter model EA80TC15 from TDK. A proper rational function with a denominator of order thirteen was used to fit the data. Since the resulting rational function was not positive real, a passive model was generated using the initial guess generation process described in section 3.1. The model was then

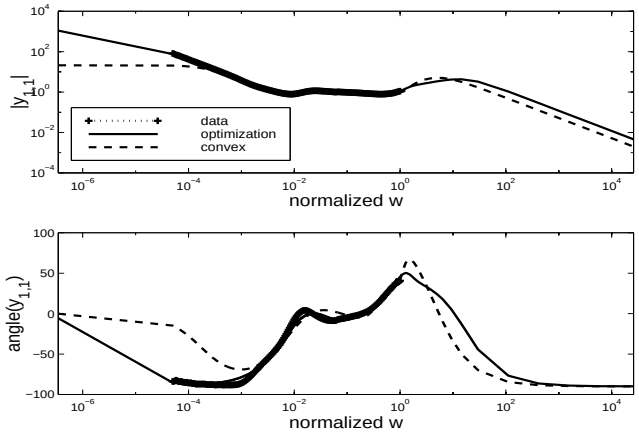

Figure 1: Fit of TDK's model F32-680 3-port filter $y_{1,1}$ function.

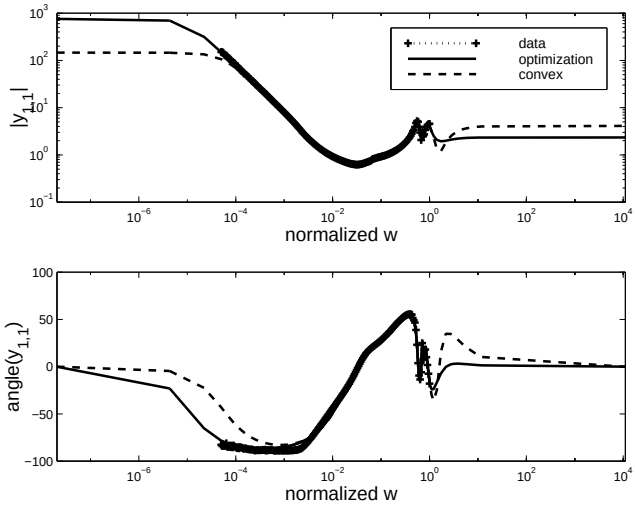

Figure 2: Fit of TDK's model EA80TC15 3-port filter $y_{1,1}$ function.

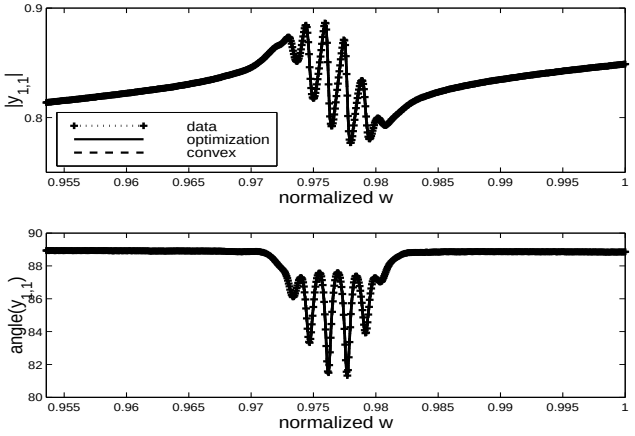

Figure 3: Fit of a SAW filter's $y_{1,1}$ function.

iteratively improved by using the proposed simultaneous numerator and denominator optimization algorithm. In Fig. 2 it can be seen that the frequency response, including the high frequency details, were accurately modeled and that the model's transfer function is indeed passive. Again, for this example, the full-approximation algorithm produces a more accurate model for the given data and model order.

Our third example uses the data set obtained for the $y_{1,1}$ parameter of a SAW filter. The data range spans a very narrow region, about 0.02 of a decade. After normalizing the frequency range, the first data point lies at 0.9536rad s^{-1}. A passive model with an order 18 numerator and denominator was obtained. Since this initial guess was already passive the iterative optimization algorithm proposed in section 3 was used directly to improve the model by simultaneously optimizing the numerator and denominator coefficients while guaranteeing the positive realness of the frequency response. The model's frequency response in the data range is represented in Fig. 3. Although it cannot be observed in the figure, the phase outside the data range stays within the $-\pi/2$ to $\pi/2$ range, thus guaranteeing passivity of the model. For this example the models produced with the proposed algorithm and the convex formulation-based algorithm are of similar accuracy.

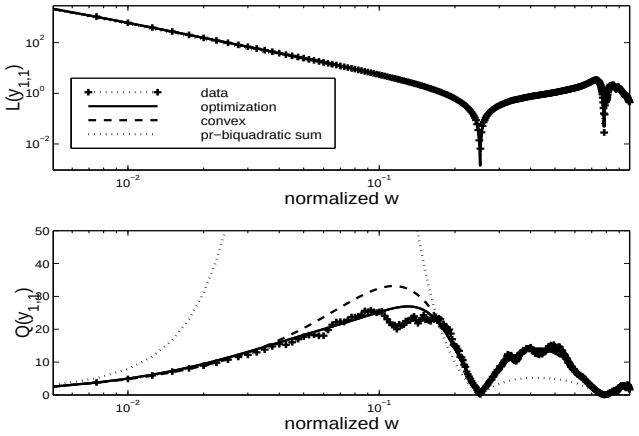

Figure 4: Fit of coil inductor's L value and Q factor.

For our last example, an s-parameter characterization of a coil inductor was obtained by using a field solver. The data spans several decades, has a resonance, and some high frequency detail, which makes the modeling task challenging. A model with an order 15 denominator and numerator was generated. A positive real initial guess was then generated by keeping the stabilized denominator and recalculating the numerator. Once that task was performed, the positive real model thus obtained was then iteratively improved by simultaneously optimizing the numerator and denominator coefficients as proposed in section 3. In Fig. 4 we illustrate that important design parameters such as the quality factor, Q,defined as $\mathrm{Im}\left[y_{1,1}\right]/\mathrm{Re}\left[y_{1,1}\right]$, which is typically somewhat difficult to fit, as well as the inductance L, defined as $\mathrm{Im}\left[y_{1,1}\right]/\omega$, are accurately modeled. For this example, besides showing the data points, the results of using the full optimization procedure and the convex approximation model, we also show the results obtained with a model derived from decomposing the transfer function into sums of positive-real functions (cf. 3.1). As can be seen from the plots, even though the fit for L is quite accurate, the result for Q shows a large error and indicates that such modeling procedures, while simple, are indeed not adequate.

5.2 On computational efficiency and accuracy

It is quite difficult to talk about the computational cost of applying the proposed algorithm and to compare it to other alternatives, namely the algorithm in [7]. The proposed algorithm is based on constructing a grid with a number of points that is linear with the model order. Therefore in theory it has the advantage of being more efficient than the convex programming based approach in [7] where the number of optimization variables grows quadratically with model order. However, the computational cost of the full optimization algorithm, as well as the accuracy of the models, is highly dependent on the initial point used for the optimization. In contrast, the convex approximation does not depend so critically on the choice of initial point. For the examples shown, the computational cost (measured in CPU time) of both approaches is generally comparable even though in some cases the full optimization method was somewhat faster to generate a model. However, by changing its initial point, it is quite easy to obtain cases where the opposite happens.

6. CONCLUSIONS

In this paper a scalar, positive real constrained, rational approximation using Theorem 1 and a grid generation and updating algorithm was proposed and tested using challenging real world examples. It was found that the generated models were indeed positive real and that the quality of the approximations obtained was good.
In comparing the proposed algorithm with the work presented in [7], the algorithm discussed here has the important advantage of allowing the denominator, as well as the numerator, to be adjusted. In some of the example data sets discussed, we have seen this to lead to better overall approximations. However, since it simultaneously optimizes the numerator and denominator coefficients, the algorithm does not share the global convergence and optimality properties associated with the convex programming approach and is thus dependent on the initial point selection. On the other hand, the proposed algorithm may be used to improve the results obtained with the algorithm presented in [7]. As future work we intend to address the issue of MIMO positive real rational approximation.

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