Delay Analysis of the Distributed RC Line

Vasant B. Rao
IBM EDA Laboratory, B306/Z3A1
Route 52, Hopewell Junction, NY 12533
raov@vnet.ibm.com

Abstract - This paper reviews the step-response of the semi-infinite distributed RC line and focuses mainly on the step-response of a finite-length RC line with a capacitive load termination, which is the most common model for a wire inside the present day integrated CMOS chips. In particular, we obtain the values of some of the common threshold-crossing times at the output of such a line and show that even the simplest first order lumped Π-approximation to the finite-length RC line terminated with a capacitive load is good enough for obtaining the 50% and 63.2% threshold-crossing times of the step-response. Higher order lumped approximations are necessary for more accurate predictions of the 10% and 90% threshold-crossing times.

1 Introduction

The distributed RC line is used to model a wire inside present day integrated CMOS chips. Consider a distributed RC line as shown in Figure 1. The underlying partial differential equation (PDE) capturing the behavior of such a line is the well-known *diffusion equation* of the form

$$rc\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} \tag{1}$$

where r and c are the *uniform* resistance and capacitance per unit length of the line and $v(x,t)$ is the voltage and position x along the line and at time t. The current $i(x,t)$ at position x along the line and at time t also satisfies the same PDE (1) above and is related to the voltage $v(x,t)$ as follows:

$$i = -\frac{1}{r}\frac{\partial v}{\partial x} \tag{2}$$

The PDE (1) can be solved in the time-domain under some special initial and boundary conditions. For example, consider the semi-infinite line starting at $x = 0$ and extending to ∞ on the right as shown in Figure 1(a). The typical initial and boundary conditions of interest in this case are:

$$\text{BC1}: v(0,t) = v_{in}(t) \text{ for all } t > 0 \tag{3}$$

$$\text{BC2}: v(\infty,t) = 0 \text{ for all } t > 0 \tag{4}$$

$$\text{IC}: v(x,0) = 0 \text{ for all } x > 0 \tag{5}$$

32nd ACM/IEEE Design Automation Conference ®

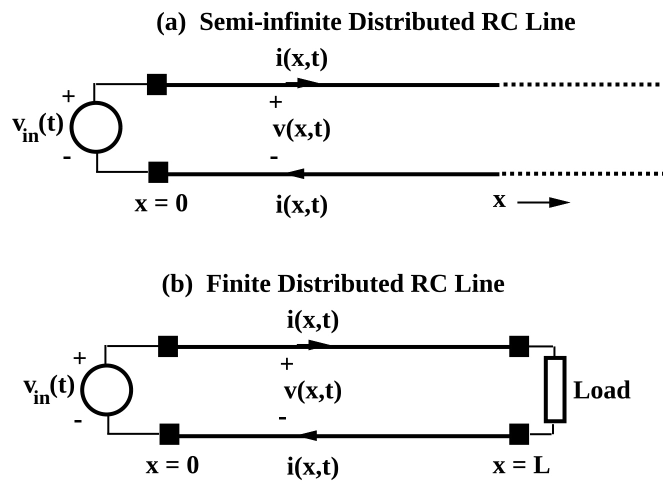

Figure 1: The Distributed RC Line

For the finite line of length L as shown in Figure 1(b), the first boundary condition BC1 and the initial condition IC (for $0 < x < L$) above usually apply, while the second boundary condition BC2 is altered to capture the effect of the *Load* applied at the end of the line $x = L$. Under the special case of an *open-ended* finite-line, the current at the end of the line should be zero, i.e., $i(L,t) = 0$ for all time $t > 0$ which on using (2) results in:

$$\text{BC3}: \frac{\partial v}{\partial x}(L,t) = 0 \text{ for all } t > 0 \tag{6}$$

Over the past several years, several authors [1-3,5-10] have solved (either completely, or approximately) the open-ended finite distributed RC line under unit-step excitation, i.e., they have obtained solutions $v(x,t)$ that solve the PDE (1) under BC1, BC3, and IC above, with $v_{in}(t) = u(t)$, where $u(t)$ is the unit-step function which is 1 for $t > 0$ and 0 for $t < 0$. A fairly detailed review of previous results is given in [1]. However, in [1], the authors incorrectly claim that

$$v(x,t) = erfc(x\sqrt{\frac{rc}{4t}}) \tag{7}$$

solves the open-ended finite distributed RC line under unit-step excitation, where the *complementary error-function* is defined to be

$$erfc(z) = 1 - \frac{2}{\sqrt{\pi}}\int_0^z e^{-y^2}dy \tag{8}$$

Clearly, $v(x,t)$ given by (7) solves the PDE (1) and satisfies BC1 and IC above but does not satisfy BC3 above. In fact it satisfies BC2 above and is therefore the solution of the semi-infinite line under unit-step excitation. Setting $x = L$ in (7) we get

$$v(L,t) = erfc(\sqrt{\frac{RC}{4t}}) \tag{9}$$

where $R = rL$ and $C = cL$ are the total resistance and capacitance of a line of length L. The authors of [1] had used (9) to predict rather large values of the various threshold crossing times of an open-ended RC line of length L, and it was this error that primarily motivated the present work. The authors of [1] realized that the error was due to incorrect cancellations in their analysis of reflections for the open-ended case [12], and their corrected results will appear in a future publication [10].

The rest of this paper is organized as follows. In Section 2, we will consider the semi-infinite RC line and in Section 3, the finite RC line terminated at $x = L$ by a general load. We will use Laplace transforms to carry out the analysis in both cases. Special cases when the input excitation is a unit-step and load is an open-circuit and when the load is a pure (lumped) capacitance will be considered in Sections 3.1 and 3.2, respectively. In Section 4, we will present some results, focusing mainly on Section 3.2. Finally, Section 5 concludes this paper.

2 The Semi-Infinite RC Line

Consider the semi-infinite Distributed RC line shown in Figure 1(a), uniform resistance per unit length r, uniform capacitance per unit length c, and whose voltage $v(x,t)$ and current $i(x,t)$ satisfy the diffusion equation (1), and $v(x,t)$ satisfies BC1, BC2, and IC given by (3), (4), and (5), respectively. Define the Laplace transform of the voltage function to be

$$V(x,s) = \int_0^\infty v(x,t)\, e^{st}\, dt \tag{10}$$

where s is the complex frequency variable. Taking the Laplace transform of (1) and applying the initial condition IC given by (5) results in

$$\frac{\partial^2 V}{\partial x^2}(x,s) = rcsV(x,s) \tag{11}$$

solving which gives the general solution in the Laplace transform domain to be

$$V(x,s) = A(s)e^{x\sqrt{rcs}} + B(s)e^{-x\sqrt{rcs}} \tag{12}$$

where $A(s)$ and $B(s)$ are independent of x and have to determined from the boundary conditions. Let $V_{in}(s)$ denote the Laplace transform of the input excitation $v_{in}(t)$. Note that BC2 given by (4) implies that $V(x,s) \to 0$ as $x \to \infty$ for all s; hence, $A(s) = 0$. Moreover, BC1 given by (3) implies $V(0,s) = V_{in}(s)$; hence, $B(s) = V_{in}(s)$. Therefore,

$$V(x,s) = V_{in}(s)e^{-x\sqrt{rcs}} \tag{13}$$

If we now invert the above transform, we can obtain the time-domain response for $v(x,t)$. To this end consider the Laplace transform pair [4]:

$$\frac{1}{2\sqrt{\pi t^3}}\, e^{-\frac{1}{4t}} \longleftrightarrow e^{-\sqrt{s}} \tag{14}$$

Using Frequency scaling property of Laplace transforms we get

$$\frac{x\sqrt{rc}}{2\sqrt{\pi t^3}}\, e^{-\frac{x^2 rc}{4t}} \longleftrightarrow e^{-x\sqrt{rcs}} \tag{15}$$

Therefore, $v(x,t)$ is the convolution of the above time-function with the input excitation $v_{in}(t)$ resulting in the general solution to the semi-infinite Distributed RC line as

$$v(x,t) = \int_0^t v_{in}(t-\tau)\frac{x\sqrt{rc}}{2\sqrt{\pi\tau^3}}\, e^{-\frac{x^2 rc}{4\tau}}\, d\tau \tag{16}$$

In general, the above convolution may be hard to evaluate. But under the special case of unit-step excitation, we have $v_{in}(t) = u(t)$ which results in

$$v(x,t) = \int_0^t \frac{x\sqrt{rc}}{2\sqrt{\pi\tau^3}}\, e^{-\frac{x^2 rc}{4\tau}}\, d\tau \tag{17}$$

Using the variable substitution $\sigma = x\sqrt{rc/4\tau}$ in (17) we get

$$v(x,t) = erfc(x\sqrt{\frac{rc}{4t}}) \tag{18}$$

as the unit-step voltage response of the semi-infinite RC line at any position $x > 0$ and time $t > 0$. One can easily verify that (18) solves the PDE (1) and satisfies all three conditions (3), (4), and (5) for the unit-step excitation.

3 The Finite RC Line

Now consider the finite Distributed RC line of length L as shown in Figure 1(b). Let $I(x,s)$ denote the Laplace transform of the current function $i(x,t)$ of the line. In the case of a general load at $x = L$, the boundary condition is best expressed in the Laplace transform domain as

$$\text{BC4}: \frac{I(L,s)}{V(L,s)} = Y_L(s) \tag{19}$$

where $Y_L(s)$ represents the admittance of the general load. Note that if the line is open at $x = L$, then $Y_L(s) = 0$ (or $i(L,t) = 0$ for all $t > 0$), while if the the line is terminated by a lumped capacitor C_L then $Y_L(s) = sC_L$ which will be the two special cases considered later. Since our domain of interest is only $0 \le x \le L$, we will restrict the initial condition to this domain and re-express the IC as

$$\text{IC}: v(x,0) = 0 \text{ for all } 0 < x \le L \tag{20}$$

Note that we have purposely omitted $x = 0$ from the IC above so that we can handle cases when the input excitation $v_{in}(t)$ has a discontinuity at $t = 0$ (e.g, the unit step function $u(t)$).

For now, we are looking for solutions $v(x,t)$ and $i(x,t)$ that satisfy (1) and (2), such that the voltage $v(x,t)$ satisfies BC1 given by (3) and IC given by (20) and the corresponding Laplace transforms satisfy BC4 given by (19). Our approach to a time-domain solution here is to find a general expression for $V(x,s)$ that satisfies BC1, IC, and BC4. We will then invert the transform under special cases. To this end, note that

$$V(x,s) = A(s)e^{x\sqrt{rcs}} + B(s)e^{-x\sqrt{rcs}} \tag{21}$$

is the general solution of the Laplace transform of the PDE (1) as in Section 2. Taking the Laplace transform of (2) we get

$$I(x,s)=\sqrt{\frac{sc}{r}}\,(B(s)e^{-x\sqrt{rcs}}-A(s)e^{x\sqrt{rcs}}) \tag{22}$$

Now, BC1 forces $V(0,s)=A(s)+B(s)=V_{in}(s)$, and BC4 gives

$$\sqrt{\frac{sC}{R}}\,\frac{(B(s)e^{-\sqrt{sRC}}-A(s)e^{\sqrt{sRC}})}{(B(s)e^{-\sqrt{sRC}}+A(s)e^{\sqrt{sRC}})}=Y_L(s) \tag{23}$$

where $R=rL$ and $C=cL$ are the total resistance and capacitance of the line. Solving for $A(s)$ and $B(s)$ and plugging these back into (21) and simplifying gives

$$V(x,s)=V_{in}(s)\frac{N(s)}{D(s)} \tag{24}$$

where

$$\begin{aligned} N(s) &= \sqrt{\frac{sC}{R}}\cosh((1-x/L)\sqrt{sRC}) \\ &+ Y_L(s)\sinh((1-x/L)\sqrt{sRC}) \end{aligned}$$

$$D(s)=\sqrt{\frac{sC}{R}}\cosh(\sqrt{sRC})+Y_L(s)\sinh(\sqrt{sRC})$$

Equation (24) is about the extent to which we can remain in the frequency domain. A similar result appears in [9]. To invert the transform, we need to consider specific input excitations $V_{in}(s)$ and loads $Y_L(s)$. Two examples are presented in the following sub-sections.

3.1 The Open-Ended Finite RC Line

Suppose $v_{in}(t)=u(t)$ and the terminating load is an open-circuit, i.e., $i(L,t)=0$ for all $t>0$. Then,

$$V_{in}(s)=\frac{1}{s}\text{ and }Y_L(s)=0 \tag{25}$$

Applying (25) to (24) we get

$$V(x,s)=\frac{1}{s}\frac{\cosh((1-x/L)\sqrt{sRC})}{\cosh(\sqrt{sRC})} \tag{26}$$

The above Laplace transform has a simple pole at $s_0=0$ and an infinite number of simple poles (roots of $\cosh(\sqrt{sRC})=0$) at $s_k=-p_k/(RC)$ for each integer $k\geq 1$, where

$$p_k=\frac{(2k-1)^2\pi^2}{4}\text{ for each }k\geq 1 \tag{27}$$

The residues corresponding to each of these poles are $\rho_0=1$ and

$$\rho_k(x)=\frac{-4\cos((1-x/L)(2k-1)\pi/2)}{(2k-1)\pi\sin((2k-1)\pi/2)}\text{ for each }k\geq 1 \tag{28}$$

Hence, the time-domain solution is

$$v(x,t)=1+\sum_{k=1}^{\infty}\rho_k(x)e^{s_k t} \tag{29}$$

which on simplifying (i.e., using $\sin((2k-1)\pi/2)=(-1)^{k+1}$ for each integer $k\geq 1$ and other manipulations) results in

$$v(x,t)=1-\sum_{k=1}^{\infty}\frac{4\sin((k-1/2)\pi x/L)}{(2k-1)\pi}e^{-\frac{(2k-1)^2\pi^2 t}{4RC}} \tag{30}$$

which is the complete solution to the unit-step response of the open-ended finite RC Line at any position $0\leq x\leq L$ and any time $t>0$. At $x=0$ we have each term in the series vanishing resulting in $v(0,t)=1$ for all time $t>0$ and this verifies BC1 under the unit-step excitation. To get an expression for the current through the line, we use (2) and (30) to get

$$i(x,t)=\frac{2}{R}\sum_{k=1}^{\infty}\cos((k-1/2)\pi x/L)e^{-\frac{(2k-1)^2\pi^2 t}{4RC}} \tag{31}$$

Plugging in $x=L$ above gives $i(L,t)=0$ for all time $t>0$ because $\cos((k-1/2)\pi)=0$ for all integers $k\geq 1$. This verifies the second boundary condition since the line is open-ended. Finally, the IC (20) is verified by noting that

$$v(x,0)=1-\sum_{k=1}^{\infty}\frac{4}{(2k-1)\pi}\sin((k-1/2)\pi x/L) \tag{32}$$

But the infinite series in (32) is simply the Fourier series of a periodic function in x with period $4L$ with value 1 for $0<x<2L$ and value -1 for $2L<x<4L$. Hence, $v(x,0)=0$ for all $0<x\leq L$, thus verifying the IC.

The output voltage at the end of the line is obtained by substituting $x=L$ in (30) and one obtains the expression derived by other authors such as [5]. If one takes only the first two terms of the infinite series in this case, the expression reduces to the one derived in [7].

An alternate approach followed by [6,9] is to invert the Laplace transform (26) differently. In this case we expand

$$\frac{1}{\cosh(\sqrt{sRC})}=2\sum_{n=0}^{\infty}(-1)^n e^{-(2n+1)\sqrt{sRC}} \tag{33}$$

Therefore (26) can be re-written as

$$V(x,s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{s}(e^{-(2n+x/L)\sqrt{sRC}}+e^{-(2n+2-x/L)\sqrt{sRC}}) \tag{34}$$

The above expression can be inverted by using the Laplace transform-pair (15) and convolution as in Section 2, to get

$$\begin{aligned} v(x,t) &= \sum_{n=0}^{\infty}(-1)^n\left\{erfc((2n+x/L)\sqrt{\frac{RC}{4t}})\right. \\ &+ \left. erfc((2n+2-x/L)\sqrt{\frac{RC}{4t}})\right\} \end{aligned} \tag{35}$$

It can be verified (by evaluating through a computer program) that the two equations (30) and (35) result in identical values for the same x and t. The only difference is that the series in (30) converges faster for large values of $t>0$ while the series in (35) converges faster for small values of $t>0$. Choosing a threshold value of $\epsilon=10^{-20}$ in deciding convergence, and

Table 1: The first 10 positive roots p_k of (39) versus $a = C_L/C$

k	$a = 0.0$	$a = 0.5$	$a = 1.0$	$a = 5.0$	$a = 10.0$
1	2.4674	1.1597	0.7402	0.1874	0.0968
2	22.2066	13.2758	11.7349	10.2652	10.0685
3	61.6850	43.2745	41.4388	39.8773	39.6782
4	120.9027	92.7284	90.8082	89.2259	89.0263
5	199.8595	161.8569	159.9033	158.3134	158.1136
6	298.5555	250.7032	248.7334	247.1399	246.9401
7	416.9908	359.2800	357.3011	355.7056	355.5057
8	555.1652	487.5916	485.6072	484.0105	483.8106
9	713.0789	635.6401	633.6520	632.0546	631.8547
10	890.7318	803.4264	801.4359	799.8379	799.6379

working with double precision, at most 4 terms are needed for convergence using the series in (30) for $t > 0.4RC$ while at most 4 terms are needed for convergence using the series in (35) for $t < 0.4RC$. One can then use this rule to pick the appropriate series to evaluate the time-domain solution $v(x,t)$ for the open-ended finite RC line.

3.2 The RC Line Terminated by a Capacitance

Suppose $v_{in}(t) = u(t)$ and the terminating load is a pure capacitance C_L. Then,

$$V_{in}(s) = \frac{1}{s} \text{ and } Y_L(s) = sC_L \tag{36}$$

Applying (36) to (24) we get

$$V(x,s) = \frac{1}{s}\frac{N(s)}{D(s)} \tag{37}$$

where

$$N(s) = \cosh((1-x/L)\sqrt{sRC}) + a\sqrt{sRC}\sinh((1-x/L)\sqrt{sRC})$$

$$D(s) = \cosh(\sqrt{sRC}) + a\sqrt{sRC}\sinh(\sqrt{sRC})$$

and

$$a = \frac{C_L}{C} \tag{38}$$

is the ratio of the load capacitance to the total capacitance of the line. This real parameter $a \geq 0$ will play a major role in determining the time-domain solution $v(x,t)$. In particular, note that the above Laplace transform has a simple pole at $s_0 = 0$ and an infinite number of simple poles (roots of $D(s) = 0$) at $s_k = -p_k/(RC)$ for each integer $k \geq 1$, where p_k is the k-th *positive root* of the following transcendental equation in a real variable y

$$\cos(\sqrt{y}) - a\sqrt{y}\sin(\sqrt{y}) = 0 \tag{39}$$

For a given value of the real parameter a, the above equation will have infinite positive roots with the k-th positive root p_k lying between $(k-1)^2\pi^2$ and $(k-1/2)^2\pi^2$ for each integer $k \geq 1$, and can be found by applying say the Newton-Raphson technique. To start the Newton-Raphson iterations, one can pick $y_0 = 2/(2a+1)$ for $k = 1$, or $y_0 = (k-1)^2\pi^2 + 2/a$ for $k > 1$, and the iterations usually converge to p_k within 4 or 5 iterations. Table 1 provides the first 10 positive roots for a few values of $a \geq 0$. Note that $a = 0$ corresponds to $C_L = 0$ which means that the load is an open-circuit; hence, the values of p_k correspond to Equation (27) in Section 3.1.

The residues corresponding to each of these poles are $\rho_0 = 1$ and

$$\rho_k(x) = \frac{2(a\sqrt{p_k}\sin((1-x/L)\sqrt{p_k}) - \cos((1-x/L)\sqrt{p_k}))}{\sqrt{p_k}((1+a)\sin(\sqrt{p_k}) + a\sqrt{p_k}\cos(\sqrt{p_k}))} \tag{40}$$

for each integer $k \geq 1$. Hence, the time-domain solution is

$$v(x,t) = 1 + \sum_{k=1}^{\infty} \rho_k(x) e^{-\frac{p_k t}{RC}} \tag{41}$$

Note that at $x = 0$, the residues given by (40) evaluate to 0 since p_k solves (39) for each $k \geq 1$. Hence $v(0,t) = 1$ which satisfies BC1 for the unit-step excitation. At the end of the line, $x = L$, the residues become

$$\rho_k(L) = \frac{-2}{\sqrt{p_k}((1+a)\sin(\sqrt{p_k}) + a\sqrt{p_k}\cos(\sqrt{p_k}))} \tag{42}$$

for each integer $k \geq 1$. resulting in

$$v(L,t) = 1 - \sum_{k=1}^{\infty} \frac{2\, e^{-\frac{p_k t}{RC}}}{\sqrt{p_k}((1+a)\sin(\sqrt{p_k}) + a\sqrt{p_k}\cos(\sqrt{p_k}))} \tag{43}$$

which is the solution for the voltage at the end of the line. The fact that this solution satisfies the second boundary condition

$$C_L \frac{\partial v}{\partial t}(L,t) = -\frac{1}{r}\frac{\partial v}{\partial x}(L,t) \tag{44}$$

can be easily verified. For a given $a > 0$ and $0 < x < L$, the series in (41) converges fairly quickly for large values of t but very slowly for small values of t approaching 0. Due to the finite precision of the computer one gets into numerical convergence problems for extremely small values of t. However, if a sufficiently large number of terms is evaluated in the series, and double precision is used, then one can see that the values of $v(x,t)$ become smaller and smaller as the value of t approaches 0, thereby, verifying the IC.

Table 2: A comparison of threshold-crossing times

$a = C_L/C$	50% crossing times			63.2% crossing times		
	Dist.	Π_2	Π_1	Dist.	Π_2	Π_1
0.0	0.379 RC	0.375 RC	0.347 RC	0.503 RC	0.507 RC	0.500 RC
0.5	0.739 RC	0.732 RC	0.693 RC	1.004 RC	1.007 RC	1.000 RC
1.0	1.089 RC	1.079 RC	1.040 RC	1.503 RC	1.505 RC	1.500 RC
5.0	3.863 RC	3.851 RC	3.812 RC	5.501 RC	5.501 RC	5.500 RC
10.0	7.329 RC	7.317 RC	7.278 RC	10.501 RC	10.501 RC	10.500 RC
	10% crossing times			90% crossing times		
$a = C_L/C$	Dist.	Π_2	Π_1	Dist.	Π_2	Π_1
0.0	0.130 RC	0.101 RC	0.053 RC	1.031 RC	1.063 RC	1.151 RC
0.5	0.220 RC	0.183 RC	0.105 RC	2.127 RC	2.173 RC	2.303 RC
1.0	0.287 RC	0.249 RC	0.158 RC	3.263 RC	3.312 RC	3.454 RC
5.0	0.726 RC	0.690 RC	0.579 RC	12.454 RC	12.507 RC	12.664 RC
10.0	1.254 RC	1.217 RC	1.106 RC	23.963 RC	24.017 RC	24.177 RC

4 Results

We now present some results on unit-step response of the finite RC line of length L, with total resistance $R = rL$, total capacitance $C = cL$, and terminated with a load capacitance $C_L = aC$ as studied in Section 3.2. Note that the case $a = 0.0$ corresponds to an open-ended line studied in Section 3.1. Consider the distributed RC line as shown in Figure 2(a). The simplest lumped approximation to the distributed RC line is to use a first-order Π_1 approximation by modeling the line as a lumped resistor R between the input and output node and a lumped capacitance of $C/2$ from either node to ground as shown in Figure 2(b). We also consider a second-order Π_2 approximation as shown in Figure 2(c), wherein the RC line is split into two halves, and each half is approximated by a first-order Π_1 approximation. Similarly, one can think of higher-order lumped Π_n approximations by splitting the RC line into n pieces and replacing each piece by a first-order Π_1 approximation. The waveforms at the output node in all three cases under unit-step input excitation are shown in Figures 3, 4, and 5, for different capacitive loads, namely, $a = 0.0$ (the open-ended line), $a = 1.0$, and $a = 10.0$, respectively. In each case, the solid line gives the waveform of the distributed line obtained using Equation (43) for $v(L,t)$ (with the p_k's computed using the Newton Raphson technique discussed in Section 3.2 for Equation (39)). The dotted line is the output waveform of the first-order Π_1 network of Figure 2(b) given by

$$v_{\pi,1}(t) = 1 - e^{-\frac{t}{RC(0.5+a)}} \tag{45}$$

Note that the lumped capacitance of $C/2$ at the source-end of this network plays no role in this response since the input is an ideal unit-step and we have considered no source impedance in this analysis. The dashed line is the output waveform of the second-order Π_2 network of Figure 2(c) given by

$$v_{\pi,2}(t) = 1 + \frac{\lambda_2}{\lambda_1 - \lambda_2} e^{-\frac{\lambda_1 t}{RC}} - \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{-\frac{\lambda_2 t}{RC}} \tag{46}$$

where λ_1 and λ_2 are the eigenvalues of the second-order system given by

$$\lambda_1 = \frac{-8(2a+1) + 4\sqrt{(4a+1)^2+1}}{4a+1} \tag{47}$$

$$\lambda_2 = \frac{-8(2a+1) - 4\sqrt{(4a+1)^2+1}}{4a+1} \tag{48}$$

Once again, note that the lumped capacitance of $C/4$ at the source-end of this network has no effect on the response since the source impedance is ignored.

It is clear from these plots, that the Π_2 network is better than the Π_1 network. Moreover, these approximations get better and better for larger loads, i.e., for larger values of a. Note that for $a = 10$, all three waveforms are almost identical. In Table 2, we present some common threshold-crossing times of the output waveforms of the distributed RC line (Equation (43)), the second-order Π_2 network (Equation (46)), and the first-order Π_1 network (Equation (45)), for different values of the capacitive load controlled by the parameter a. From these tables, it is clear that the 63.2% crossing times of all three waveforms are almost identical and the error in the 50% crossing times is less than 9%. The Π_1 network over-estimates the 90% crossing time by less than 12% but under-estimates the 10% crossing time by as much as 60% as in the $a = 0.0$ (open-ended line) case. The Π_2 network, on the other hand, under-estimates the 10% crossing time by less than 22%. In any case, the errors in the threshold crossing times between the approximations and the distributed network reduces as the load C_L is increased. Finally, note that errors in the threshold-crossing times (in particular the 10% crossing time) will reduce further as higher-order lumped approximations (such as the lumped Π_n approximations for $n \geq 3$) are used to approximate the distributed RC line. In such cases one can use a tool such as RICE [11] to solve the lumped network and compute the required threshold crossing times.

5 Conclusions

This paper presents the exact analysis of a distributed RC line. Both the semi-infinite length line and the finite length line terminated by an arbitrary load are studied in the Laplace transform domain. To get the results back in the time-domain, the input excitation is assumed to be the unit-step function and the terminating load is assumed to be a pure capacitance which is typical in present day CMOS VLSI chips. While a closed form expression (in terms of the complementary error function) exists for the voltage at any point x on the semi-infinite line, the solution in the finite-length line (with capacitive load) case is an infinite series whose terms diminish in value. It is shown that the simplest first-order lumped Π-approximation of the distributed line is sufficient for predicting 50% and 63.2% delays fairly accurately. For 10% and 90% higher order approximations are needed. Moreover, the errors in the the lumped circuit approximations reduce significantly as the capacitive load increases.

References

1. A. B. Kahng and S. Muddu, "Delay Analysis of VLSI Interconnections Using the Diffusion Equation Model", *Proc. 31st Design Automation Conference*, San Diego, CA, pp. 563-569, June 1994.
2. R. J. Antinone and G. W. Brown, "The Modeling of Resistive Interconnects for Integrated Circuits", *IEEE Journal of Solid State Circuits*, Vol 18, pp. 200-203, April 1983.
3. H. B. Bakoglu, *Circuits, Interconnects, and Packaging*, Addison Wesley, 1990.
4. M. Healey, *Tables of Laplace, Heaviside, Fourier, and Z Transforms*, W & R Chambers Ltd., 1967.
5. W. M. Kaufman and S. J. Garrett, "Tapered Distributed Filters", *IRE Transactions on Circuit Theory*, pp. 329-336, December 1962.
6. H. L. Mattes, "Behavior of a Single Transmission Line Stimulated with a Step Function", *manuscript*, August 1993.
7. T. Sakurai, "Approximation of Wiring Delay in MOSFET LSI", *IEEE Journal of Solid State Circuits*, pp. 418-426, August 1983.
8. A. Wilnai, "Open-Ended RC Line Model Predicts MOSFET IC Response", *Electron Devices Newsletter*, pp. 53-54, December 15, 1971.
9. A. B. Kahng and S. Muddu, "A General Methodology for Response and Delay Computations in VLSI Interconnects", *technical report* UCLA CSD-TR-940015, April 1994.
10. A. B. Kahng and S. Muddu, "Delay Analysis of VLSI Interconnects Using the Diffusion Equation Model", manuscript submitted to *IEEE Trans. on Circuits and Systems*, May 1994 (revised February 1995).
11. C. L. Ratzlaff, N. Gopal, and L. T. Pillage, "RICE: Rapid Interconnect Circuit Evaluator", *Proc. 28th Design Automation Conference*, June 1991.
12. A. B. Kahng and S. Muddu, *personal communication*, February 1995.

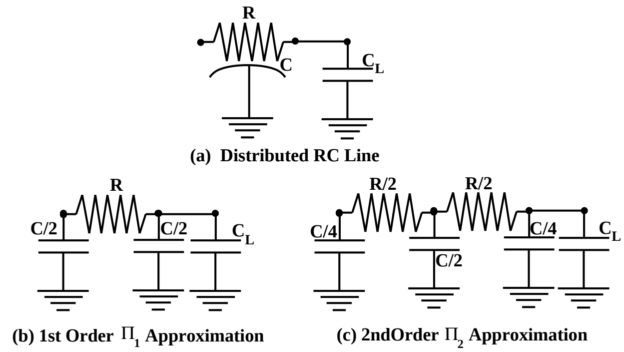

Figure 2: The RC Line Terminated by a Capacitive Load

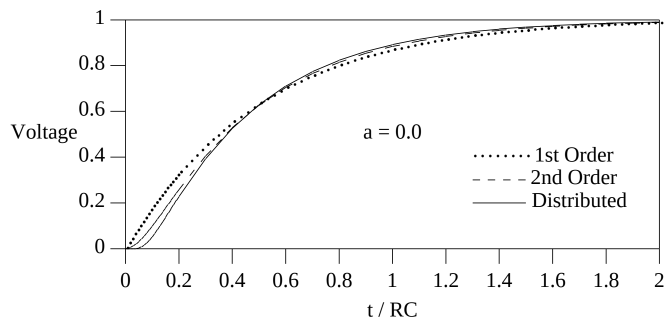

Figure 3: The Unit Step Response of the Open-Ended RC Line

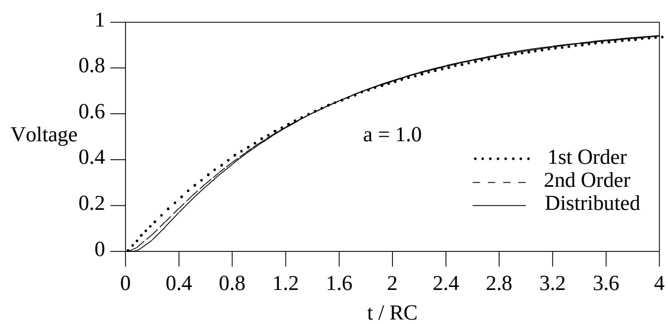

Figure 4: The Unit Step Response of the RC Line Terminated with $C_L = C$

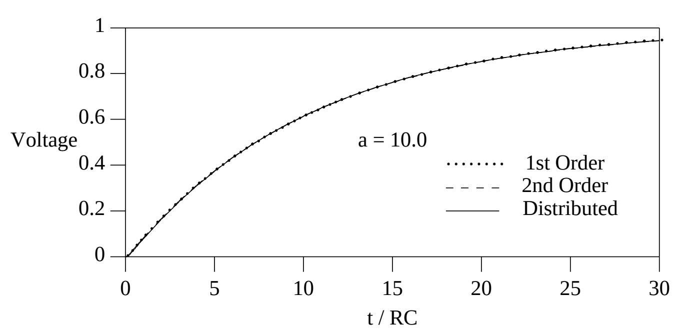

Figure 5: The Unit Step Response of the RC Line Terminated with $C_L = 10C$